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Axiomatic systems for rough sets and fuzzy rough sets

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Abstract

Rough set theory is an important tool for approximate reasoning about data. Axiomatic systems of rough sets are significant for using rough set theory in logical reasoning systems. In this paper, outer product method are used in rough set study for the first time. By this approach, we propose a unified lower approximation axiomatic system for Pawlak's rough sets and fuzzy rough sets. As the dual of axiomatic systems for lower approximation, a unified upper approximation axiomatic characterization of rough sets and fuzzy rough sets without any restriction on the cardinality of universe is also given. These rough set axiomatic systems will help to understand the structural feature of various approximate operators.

Keywords: Rough sets; Fuzzy sets; Fuzzy rough sets; Lower approximations; Upper approximations; Axioms

1. Introduction

Rough set theory [16,17] is a new approach for reasoning about data. It has achieved a large amount of real applications such as medicine, information analysis, data mining, control and linguistics. As a tool to handling imperfect data, it complements other theories that deal with uncertain data, such as probability theory, evidence theory and fuzzy set theory. The main idea of rough sets corresponds to the lower and upper approximations. Pawlak's definitions for the lower and upper approximations were originally introduced with reference to an equivalence relation. Many interesting properties of the lower and upper approximations have been derived by Pawlak [16,17] based on the equivalence relations. In this paper, we study a reverse problem. That is, can we characterize the notion of the lower and upper approximations in terms of those properties? We answer the question affirmatively.

Since classical rough set theory is based on equivalence relations, the equivalence relation is too restrictive for most applications. To address this issue, the main idea is focused on generalizing and interpreting rough sets. For example, rough set model is extended to include general binary relations [1,3,6,19,21,24,26–32,35] and coverings [2,32–34]. Some researchers even extended classical rough sets to Boolean algebras [9,13] and lattices [14,20]. Dubois and Prade [4,5] were the first one of researchers to propose the concept of fuzzy rough

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sets. They constructed a pair of lower and upper approximation operators for fuzzy sets with respect to fuzzy similarity relation by using the t-norm Min and its dual conorm Max. The axiomatic approach takes the lower and upper approximations as primitive notions. In this approach, a set of axioms is used to characterize the approximations. Lin and Liu [10] proposed six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the lower and upper approximations are the same as the abstract operators. Mi and Zhang [11] and Wu and Zhang [22] studied axioms for approximation operators in fuzzy environment. A similar problem was also investigated earlier by Morsi and Yakout [15]. The most important axiomatic studies for crisp rough sets were made by Yao [27–29] and Yao and Lin [30], in which various classes of crisp rough set algebras were characterized by different sets of axioms. By using inner product method and matrix method, Liu [12] gave a unified axiomatic system to characterizes the upper approximation operations for both Pawlak's rough sets and fuzzy rough sets with a finite universe. However, the matrix method fails when the universe is an infinite set. Besides the inner product method can only express the axiomatic systems for the upper approximation operations and but not for the lower approximation operations. In this paper, we use outer product method to study lower approximation operations for both Pawlak's rough sets and fuzzy rough sets. Especially, we express axiomatic systems for the lower approximation operations in these two rough set models. As the dual of axiomatic systems for lower approximation, similar axiomatic systems for the upper approximations are also obtained. The new expression we obtained unifies existing axiomatic systems for Pawlak's rough sets and fuzzy rough sets.

Many authors who studied axiomatic system of rough sets were based on set-theoretical methods [5,7,8,10,11,25–32,34]. However, according to the literature, none of them used characteristic function as we do in this paper. They either studied axiomatic system for Pawlak's rough sets or studied axiomatic system for fuzzy rough sets, but not both. In this paper we use the inner and outer product methods to discuss the axiomatic systems of rough sets. Furthermore, we can use the same expression to characterize the lower and upper approximations of rough sets both in case of crisp sets and in case of fuzzy sets. As we know, the outer product method is first used in rough sets. By using this approach, we obtain a unified axiomatic characterization of the lower and upper approximations for Pawlak's rough sets and fuzzy rough sets without any restriction on the cardinality of universe.

The paper is organized as follows. In Section 2, we introduce the definitions and properties of rough sets. We also define the inner and outer products of sets. In Section 3, we study axiomatic systems for Pawlak's rough set. By using the outer and inner products of sets, we propose axiomatic systems for lower and upper approximations of Pawlak's rough set over arbitrary universe, respectively. In Section 4, the definitions of fuzzy rough sets are introduced and the properties of lower and upper approximations are presented. In Section 5, we characterize the fuzzy rough set lower and upper approximations with the same two axioms. Finally, Section 6 concludes the paper.

2. Preliminaries

A binary relation is an equivalence relation iff it is reflexive, symmetric and transitive. For every equivalence relation there is a partition and vice versa. Let U be the (finite or infinite) universe of discourse. Let R be a given equivalence relation on U. The family of all equivalence classes is a set which is called the quotient set and denoted by U/R. We use $[x]$ to denote an equivalence class in R containing an element $x \in U$. For each subset X of U, we associate two subsets, lower and upper approximations [16,17]:

$$\underline{R}X = \{x|[x] \subseteq X\} \text{ and } \overline{R}X = \{x|[x] \cap X \neq \emptyset\},$$

respectively. The pair $RX(= (\underline{R}X, \overline{R}X))$ is referred to as the rough set of X. The rough set $(\underline{R}X, \overline{R}X)$ denotes the description of X under the present knowledge, i.e., the classification of U.

Pawlak's rough sets can be easily extended by considering other types of binary relations. Let $R \subseteq U \times U$ be a binary relation on U without any additional constraints. By taking the similarity class $r(x) = \{y \in U|xRy\}$ instead of the equivalence class we can obtain a generalization of the definitions of the lower approximation and the upper approximation of X by $\underline{R}X = \{x \in U|r(x) \subseteq X\}$ and $\overline{R}X = \{x \in U|r(x) \cap X \neq \emptyset\}$ [27], respectively.

A very useful concept for sets is the characteristic function. If X is a subset of a universal set U, the characteristic function of X, still denoted by X, is defined for each $x \in U$ as follows:

$$X(x) = \begin{cases} 1, & x \in X, \\ 0, & x \notin X. \end{cases}$$

We may add and multiply characteristic functions, since their values are numbers, and these operations sometimes help us prove theorems about properties of subsets of a universal set. For the arbitrary binary relation R on universal set U, $R(x,y)$ is defined by

$$R(x,y) = \begin{cases} 1, & (x,y) \in R, \\ 0, & (x,y) \notin R. \end{cases}$$

By using characteristic function, the lower and upper approximations of rough set can be rewritten as follows.

Proposition 2.1. *Let U be an arbitrary universal set, $P(U)$ the power set of U, and R an arbitrary binary relation on U. The lower and upper approximations of set $X \in P(U)$, $\underline{R}X$ and $\overline{R}X$, are subsets of U such that*

$$(\overline{R}X)(x) = \vee_{y\in U}(R(x,y) \wedge X(y)), \quad x \in U$$

and

$$(\underline{R}X)(x) = \wedge_{y\in U}((1 - R(x,y)) \vee X(y)), \quad x \in U,$$

where $\wedge = inf$ and $\vee = sup$.

Proposition 2.1 is significant since it is the reasoning foundation of this paper. In order to get the construction of the lower approximation operation of rough set over an arbitrary universal set, we need to introduce the inner and outer products of two subsets. Recall that the inner product (respectively, outer product) of two subsets X, Y of U, denoted by (X, Y) (respectively, denoted by $[X, Y]$), is defined as $(X, Y) = \vee_{x\in U}(X(x) \wedge Y(x))$ (respectively, is defined as $[X, Y] = \wedge_{x\in U}(X(x) \vee Y(x))$). The inner product (X, Y) is equal to 1 if the intersection between X and Y is not empty and it is equal to 0 otherwise, while the outer product $[X, Y]$ is equal to 1 if the union of X and Y is the universe U and it is equal to 0 otherwise.

The inner and outer products have the following important properties.

Proposition 2.2. *Let U be an arbitrary universe and $P(U)$ be the power set of U. The inner and outer products of subsets have the following properties*:

(1) $(\cup_{i\in I}X_i, Y) = \vee_{i\in I}(X_i, Y), [\cap_{i\in I}X_i, Y] = \wedge_{i\in I}[X_i, Y]$ *for any given index set* $I, X_i, Y \in P(U)$;
(2) *commutativity* $(X, Y) = (Y, X), [X, Y] = [Y, X]$ *for* $X, Y \in P(U)$;
(3) *if* $(X, Y) = (X, Z)$ *for all* $X \in P(U)$, *then* $Y = Z$;
(4) *if* $[X, Y] = [X, Z]$ *for all* $X \in P(U)$, *then* $Y = Z$;
(5) $(X, \overline{R}Y) = \vee_{x\in U}\vee_{y\in U}(X(x) \wedge R(x,y) \wedge Y(y))$;
(6) $[X, \underline{R}Y] = \wedge_{x\in U}\wedge_{y\in U}(X(x) \vee (1 - R(x,y)) \vee Y(y))$;
(7) $(X, Y)' = 1 - (X, Y) = [X', Y']$ *and,* $[X, Y]' = 1 - [X, Y] = (X', Y')$ *for all* $X, Y \in P(U)$, *where* X' *is the complement of* X.

Proof

(1) For any given index set $I, X_i \in P(U),\ i \in I$, and $Y \in P(U)$

$$\begin{aligned}(\cup_{i\in I}X_i, Y) &= \vee_{x\in U}((\cup_{i\in I}X_i)(x) \wedge Y(x)) = \vee_{x\in U}((\vee_{i\in I}X_i(x)) \wedge Y(x)) = \vee_{x\in U}\vee_{i\in I}(X_i(x) \wedge Y(x)) \\ &= \vee_{i\in I}\vee_{x\in U}(X_i(x) \wedge Y(x)) = \vee_{i\in I}(X_i, Y),\end{aligned}$$

$[\cap_{i\in I}X_i, Y] = \wedge_{i\in I}[X_i, Y]$ can be proved in a similar way.
(2) is clear.
(3) If $Y \neq Z$, then there exists some element $x \in U$ such that $Y(x) \neq Z(x)$. Let $X = \{x\}$, then $(X, Y) = Y(x) \neq Z(x) = (X, Z)$. This is a contradiction.

(4) Similar to (3).
(5) Using Proposition 2.1,

$$(X,\overline{R}Y) = \vee_{x\in U}(X(x)\wedge \overline{R}Y(x)) = \vee_{x\in U}(X(x)\wedge(\vee_{y\in U}(R(x,y)\wedge Y(y))))$$
$$= \vee_{x\in U}\vee_{y\in U}(X(x)\wedge R(x,y)\wedge Y(y)).$$

(6) Similar to (5).
(7) $(X,Y)' = 1-\vee_{x\in U}(X(x)\wedge Y(x)) = \wedge_{x\in U}(1-X(x)\wedge Y(x)) = \wedge_{x\in U}((1-X(x))\vee(1-Y(x)))$
$= \wedge_{x\in U}(X'(x)\vee Y'(x)) = [X',Y']$.

$[X,Y]' = (X',Y')$ can be proved in a similar way. □

The basic properties of the lower and upper approximations are given by the following proposition.

Proposition 2.3. *Let U be an arbitrary universal set and R be an arbitrary binary relation on U. Then for* $X,Y\subseteq U$

(1) *For any given index set I and* $X_i\in P(U)$, $\overline{R}(\cup_{i\in I}X_i) = \cup_{i\in I}\overline{R}(X_i), \underline{R}(\cap_{i\in I}X_i) = \cap_{i\in I}\underline{R}X_i$;
(2) $\overline{R}\emptyset = \emptyset, \underline{R}U = U$;
(3) $X\subseteq Y$ *implies* $\underline{R}X\subseteq\underline{R}Y, \overline{R}X\subseteq\overline{R}Y$;
(4) $\overline{R}(X\cap Y)\subseteq\overline{R}(X)\cap\overline{R}(Y), \underline{R}(X\cup Y)\supseteq\underline{R}X\cup\underline{R}Y$;
(5) $\underline{R}(X') = (\overline{R}X)', \overline{R}(X') = (\underline{R}X)'$. *Moreover, If R is an equivalence relation on U, then*
(6) $\underline{R}X\subseteq X\subseteq\overline{R}X$;
(7) $\underline{R}\underline{R}X = \underline{R}X$, *and* $\overline{R}\overline{R}X = \overline{R}X$;
(8) $(X,\overline{R}Y) = (Y,\overline{R}X)$, *and* $[X,\underline{R}Y] = [Y,\underline{R}X]$ *for all* $X,Y\in P(U)$.

Proof. Parts (1)–(7) can be found in [17] or [18].
(8) Using Proposition 2.2(5) and symmetric property of R,

$$(X,\overline{R}Y) = \vee_{x\in U}\vee_{y\in U}(X(x)\wedge R(x,y)\wedge Y(y)) = \vee_{x\in U}\vee_{y\in U}(X(x)\wedge R(y,x)\wedge Y(y)) = (Y,\overline{R}X).$$

Similarly, $[X,\underline{R}Y] = [Y,\underline{R}X]$. □

3. Axiomatization of rough sets

The axiomatic approach aims to investigate the logical structure of rough sets rather than to develop some methods for applications. This section shows that if there exists an operator L, which assigns every subset X of the universe U to a subset $L(X)$, such that the assignment satisfies two axioms, then there is an equivalence relation defined on U such that $L(X)$ is Pawlak's rough set lower approximation of X. We also study the dual problem.

We make full use of characteristic functions for sets. Note that the characteristic function of X is still denoted by X. That is, we do not distinguish the subset X of U and its corresponding characteristic function.

Lemma 3.1. [9] *Let U be an arbitrary universal set and* $P(U)$ *be the power set of U. Suppose* $L : P(U)\to P(U)$ *is a unary operator. Then there exists a unique binary relation R on U such that* $L(X) = \underline{R}X$ *for subset X of U, if and only if L satisfies the properties*:

(1) $L(U) = U$;
(2) *for any given index set I,* $L(\cap_{i\in I}X_i) = \cap_{i\in I}L(X_i)$ *for all* $X_i\subseteq U,\ i\in I$.

Note that if U is a finite set, condition (2) in Lemma 3.1 is equivalent to $L(X\cap Y) = L(X)\cap L(Y)$ for all $X,Y\in P(U)$. For the upper approximation, we have the following dual result.

Lemma 3.1′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $H : P(U) \to P(U)$ is the unary operator. There exists a unique binary relation R on U such that $H(X) = \overline{R}X$ for subset X of U, if and only if H satisfies the properties*:

(1) $H(\emptyset) = \emptyset$;
(2) *for any given index set I, $H(\cup_{i\in I}X_i) = \cup_{i\in I}H(X_i)$ for all $X_i \subseteq U$.*

Proof. The proof can be derived by duality from the ones of lower approximations and we omit it. □

Theorem 3.1. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $L : P(U) \to P(U)$ is a unary operator, L satisfies $L(U) = U$, and for any given index set I, $L(\cap_{i\in I}X_i) = \cap_{i\in I}L(X_i)$ for all $i \in I$ and $X_i \in P(U)$. We have the following three results.*

(1) *There exists a unique reflexive relation R on U such that $L(X) = \underline{R}X$ for subset X of U if and only if $L(X) \subseteq X$ for all $X \in P(U)$;*
(2) *There exists a unique symmetric relation R on U such that $L(X) = \underline{R}X$ for subset X of U if and only if $[X, L(Y)] = [Y, L(X)]$ for all $X, Y \in P(U)$;*
(3) *There exists a unique transitive relation R on U such that $L(X) = \underline{R}X$ for subset X of U if and only if $L(X)) \subseteq L(L(X)$ for all $X \in P(U)$.*

Proof. By Lemma 3.1, there exists a unique binary relation R on U such that $L(X) = \underline{R}X$ for all $X \subseteq U$. The proof of parts (1) and (3) can be found in [9]. We only need to prove part (2).(2) If R is a symmetric relation on U, then $R(x,y) = R(y,x)$ for $x, y \in U$ and

$$\begin{aligned}[X, \underline{R}Y] &= \wedge_{x\in U}\wedge_{y\in U}(X(x) \vee (1 - R(x,y)) \vee Y(y)) = \wedge_{x\in U}\wedge_{y\in U}(X(y) \vee (1 - R(y,x)) \vee Y(x)) \\ &= \wedge_{x\in U}\wedge_{y\in U}(Y(x) \vee (1 - R(x,y)) \vee X(y)) = [Y, \underline{R}X].\end{aligned}$$

Conversely, if $[X, L(Y)] = [Y, L(X)]$, then for subsets $X = \{x\}' = U - \{x\}$ and $Y = \{y\}' = U - \{y\}, x, y \in U$,

$$[X, L(Y)] = \wedge_{a\in U}\wedge_{b\in U}(X(a) \vee (1 - R(a,b)) \vee Y(b)) = 1 - R(x,y).$$

Similarly, $[Y, L(X)] = 1 - R(y,x)$, it follows $R(x,y) = R(y,x)$ for $x, y \in U$. Thus R is a symmetric relation on U. □

For the upper approximation, we have the following dual results.

Theorem 3.1′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $H : P(U) \to P(U)$ is a unary operator, H satisfies $H(\emptyset) = \emptyset$, and for any given index set I, $H(\cup_{i\in I}X_i) = \cup_{i\in I}H(X_i)$ for all $i \in I$ and $X_i \in P(U)$. We have the following three results.*

(1) *There exists a unique reflexive relation R on U such that $H(X) = \overline{R}X$ for subset X of U if and only if $X \subseteq H(X)$ for all $X \in P(U)$.*
(2) *There exists a unique symmetric relation R on U such that $H(X) = \overline{R}X$ for subset X of U if and only if $(X, H(Y)) = (Y, H(X))$ for all $X, Y \in P(U)$.*
(3) *There exists a unique transitive relation R on U such that $H(X) = \overline{R}X$ for subset X of U if and only if $H(H(X)) \subseteq H(X)$ for all $X \in P(U)$. Note that condition $[X, L(Y)] = [Y, L(X)]$ follows $L(U) = U$, and $L(\cap_{i\in I}X_i) = \cap_{i\in I}L(X_i)$ for any given index set I, i.e., we have the following result*:

Lemma 3.2. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. If $L : P(U) \to P(U)$ is the unary operator that satisfies $[X, L(Y)] = [Y, L(X)]$, then $L(U) = U$, and $L(\cap_{i\in I}X_i) = \cap_{i\in I}L(X_i)$ for any given index set I and $X_i \subseteq U$.*

Proof. For any $X \in P(U), [U, L(X)] = 1$. If $L(U) \neq U$, then there exists some $x \in (L(U))' = U - L(U)$, assume $X = \{x\}' = U - \{x\}$,

$$[X, L(U)] = \wedge_{y \in U}(X(y) \vee L(U)(y)) = X(x) \vee L(U)(x) = 0.$$

Thus $[X, L(U)] \neq [U, L(X)]$, this is a contradiction.

Since, for any given index set I,

$$[Y, L(\cap_{i\in I} X_i)] = [\cap_{i\in I} X_i, L(Y)] = \wedge_{i\in I}[X_i, L(Y)] = \wedge_{i\in I}[Y, L(X_i)] = [Y, \cap_{i\in I} L(X_i)]$$

for all $X_i, Y \subseteq U$. Thus, by Proposition 2.2(4), we have $L(\cap_{i\in I} X_i) = \cap_{i\in I} L(X_i)$. □

Lemma 3.2′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. If $H : P(U) \to P(U)$ is the unary operator that satisfies $(X, H(Y)) = (Y, H(X))$, then $H(\emptyset) = \emptyset$, and $H(\cup_{i\in I} X_i) = \cup_{i\in I} H(X_i)$ for any given index set I and $X_i \subseteq U$.*

Combining Theorem 3.1 and Lemma 3.2, we have the following interesting corollary. That is, we can characterize the lower approximation base on a symmetric binary relation R on U with a simple axiom.

Corollary 3.1. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. If $L : P(U) \to P(U)$ is the unary operator that satisfies $[X, L(Y)] = [Y, L(X)]$ for all $X, Y \in P(U)$, then there exists a unique symmetric binary relation R on U such that $L(X) = \underline{R}X$ for all $X \in P(U)$.*The following result is the dual of Corollary 3.1.

Corollary 3.1′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. If $H : P(U) \to P(U)$ is the unary operator that satisfies $(X, H(Y)) = (Y, H(X))$ for all $X, Y \in P(U)$, then there exists a unique symmetric binary relation R on U such that $H(X) = \overline{R}X$ for all $X \in P(U)$.*

From Theorem 3.1 and Lemma 3.2, we obtain the axiomatization of the lower approximation operation of Pawlak's rough set in the arbitrary universal set.

Theorem 3.2. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $L : P(U) \to P(U)$ is the unary operator that satisfies axioms*:

(1) $L(X) \subseteq X$ for all $X \in P(U)$;
(2) $[X, L(Y)] = [Y, L(X)]$ for all $X, Y \in P(U)$;
(3) $L(X) \subseteq L(L(X))$ for all $X \in P(U)$.

Then there exists a unique equivalence relation R on U such that $L(X) = \underline{R}X$ for each subset X of U. If we define the upper approximation operation using the formula, $\overline{R}X = -\underline{R}(-X)$, then $(\underline{R}X, \overline{R}X)$ is a rough set.

Theorem 3.2′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $H : P(U) \to P(U)$ is the unary operator that satisfies axioms*:

(1) $X \subseteq H(X)$ for all $X \in P(U)$;
(2) $(X, H(Y)) = (Y, H(X))$ for all $X, Y \in P(U)$;
(3) $H(H(X)) \subseteq H(X)$ for all $X \in P(U)$.

Then there exists a unique equivalence relation R on U such that $H(X) = \overline{R}X$ for each subset X of U.

If R is an equivalence relation on U, then $L(X) = \underline{R}X$ for any $X \subseteq U$ satisfies all three properties in Theorem 3.2, axioms (1), (2), and (3) in Theorem 3.2 are consistent. The following example shows that the above three properties for the lower approximation operation are necessary and independent each other. Similar conclusions hold for Theorem 3.2′.

Example 3.1. Let U be an arbitrary universal set, we consider the following cases:

(a) Let $L(X) = U$ for any $X \subseteq U$. This L satisfies all properties except (1).
(b) Consider a universe $U = \{a, b, c\}$. Let R be a binary relation on U:

$$R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}.$$

If $X = \{a, c\}, Y = \{b, c\}$ and $L(X) = \underline{R}X$, then $[X, L(Y)] = 1$, but $[Y, L(X)] = 0$. This L satisfies all properties except (2).

(c) Consider a universe $U = \{a, b, c\}$. Let R be a binary relation on U:

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$$

and $L(X) = \underline{R}X$. This L satisfies all properties except (3).

Note that conditions (1), (2), and (3) in Theorem 3.2 are equivalent to the following two conditions

(1) $[X, L(Y)] = [Y, L(X)]$ for all $X, Y \in P(U)$;
(2) $X \cap L(L(X)) = L(X)$ for all $X \in P(U)$.

Theorem 3.2 can be rewritten as follows.

Theorem 3.3. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $L : P(U) \to P(U)$ is the unary operator that satisfies axioms*:

(1) $[X, L(Y)] = [Y, L(X)]$ *for all* $X, Y \in P(U)$;
(2) $X \cap L(L(X)) = L(X)$ *for all* $X \in P(U)$. Then there exists a unique equivalence relation R on U such that $L(X) = \underline{R}X$ for each subset X of U.

Theorem 3.3′. *Let U be an arbitrary universal set and $P(U)$ be the power set of U. Suppose $H : P(U) \to P(U)$ is the unary operator that satisfies axioms*:

(1) $(X, H(Y)) = (Y, H(X))$ *for all* $X, Y \in P(U)$;
(2) $X \cup H(H(X)) = H(X)$ *for all* $X \in P(U)$.

Then there exists a unique equivalence relation R on U such that $H(X) = \overline{R}X$ for each subset X of U.

Theorem 3.3, in fact, gives an axiomatic system with two simple axioms for Pawlak's rough set lower approximation operations. Theorem 3.3′ gives an axiomatic system for Pawlak's rough set upper approximation operations.

4. Fuzzy rough sets

The notion of fuzzy sets provides a convenient tool for representing vague concepts by allowing partial memberships. In this section, we define rough sets in a fuzzy approximation space over an arbitrary universe. A fuzzy set X of U is defined by a membership function: $X : U \to [0, 1], X(x), x \in U$, giving the degree of membership of x in X. Let $F(U)$ denote the fuzzy power set of U, i.e., the set of all functions from U to [0, 1]. Fuzzy set intersection, union and complement operators are defined component-wise as

(1) $(X \cap Y)(x) = X(x) \wedge Y(x)$ for $X, Y \in F(U)$;
(2) $(X \cup Y)(x) = X(x) \vee Y(x)$ for $X, Y \in F(U)$;
(3) $X'(x) = 1 - X(x)$ for $X \in F(U)$.

Where $\wedge$ denotes minimum and $\vee$ maximum. We recall that a fuzzy relation on U is a fuzzy subset of $U \times U$. If R is a fuzzy relation on U, the pair (U, R) is called a fuzzy approximation space. A fuzzy relation R on U is called a fuzzy similarity relation if R is reflexive ($R(x, x) = 1$), symmetric ($R(x, y) = R(y, x)$) and transitive ($R(x, y) \geqslant \vee_{z \in U}(R(x, z) \wedge R(z, y))$).

The concept of fuzzy rough set was proposed by Pei [18]. His idea was as follows. Let U be an arbitrary universal set and R be a fuzzy relation on U, $F(U)$ be the fuzzy power set of U. A fuzzy rough set is a pair $(\underline{R}X, \overline{R}X)$ of fuzzy sets on U such that for every $x \in U$

$$(\underline{R}X)(x) = \wedge_{y \in U}((1 - R(x, y)) \vee X(y)) \quad \text{and} \quad (\overline{R}X)(x) = \vee_{y \in U}(R(x, y) \wedge X(y)).$$

$\underline{R}X$ and $\overline{R}X$ are referred to as the lower and the upper approximation of a fuzzy set X in U, respectively. In particular, if R is a crisp binary equivalence relation on U and X is a crisp subset of U, by using Proposition 2.2, the fuzzy rough set is degenerated to Pawlak's rough sets.

The following proposition reflects the relationship between operators $\underline{R}$ and $\overline{R}$ and it also shows the properties of operators $\underline{R}$ and $\overline{R}$.

Proposition 4.1. *Let U be an arbitrary universal set and R be a fuzzy relation on U. Then for $X, Y \in F(U)$*

(1) *for any given index set I, $\overline{R}(\cup_{i\in I}X_i) = \cup_{i\in I}\overline{R}(X_i), \underline{R}(\cap_{i\in I}X_i) = \cap_{i\in I}\underline{R}X_i$ for $X_i \in F(U)$;*
(2) $\overline{R}\emptyset = \emptyset, \underline{R}U = U$;
(3) *$X \subseteq Y$ implies $\underline{R}X \subseteq \underline{R}Y, \overline{R}X \subseteq \overline{R}Y$;*
(4) $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y), \underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y$;
(5) *$\underline{R}(X') = (\overline{R}X)', \overline{R}(X') = (\underline{R}X)'$, where X' is the complement of X; Moreover, if R is a fuzzy similarity relation on U, then*
(6) $\underline{R}X \subseteq X \subseteq \overline{R}X$;
(7) *$\underline{R}\underline{R}X = \underline{R}X$ and $\overline{R}\overline{R}X = \overline{R}X$;*
(8) *$(X, \overline{R}Y) = (Y, \overline{R}X)$, and $[X, \underline{R}Y] = [Y, \underline{R}X]$.*

Proof. The proof of parts (1)–(7) can be found in [11] and [18]. The proof of part (8) is analogous to the proof of Proposition 2.3 (8) and we omit it. □

For the fuzzy sets as in Section 2, we can define the inner product (respectively, outer product) of two fuzzy sets X, Y of U, denoted by (X, Y) (respectively, denoted by $[X, Y]$), as $(X, Y) = \vee_{x\in U}(X(x) \wedge Y(x))$ (respectively, $[X, Y] = \wedge_{x\in U}(X(x) \vee Y(x))$). Similar properties of the inner and outer products can be obtained.

5. An axiomatic system of fuzzy rough sets

In this section we will show that the axiomatic system in Theorem 3.2 also characterizes the lower approximation operator of fuzzy rough sets over an arbitrary universe. Let U be the universal set, with $\lambda \in [0, 1]$, $X \in F(U)$, fuzzy sets λX and $\lambda \vee X$ in U are defined as follows:

$$(\lambda X)(x) = \lambda \wedge X(x), \quad (\lambda \vee X)(x) = \lambda \vee X(x),$$

for all $x \in U$. For subset X of U, $X' = U - X$ denotes the complement of X. Note that for any given $x \in U$, $\{x\}'(y)$ is equal to 0 if $y = x$ and it is equal to 1 otherwise. The following lemma is well known:

Lemma 5.1. *Any $X \in F(U)$ can be written as*

$$X = \cup_{x\in U}X(x)\{x\} = \cap_{x\in U}(X(x) \vee \{x\}').$$

Next, we will consider the abstract fuzzy set-theoretic operator $L : F(U) \to F(U)$.

Lemma 5.2. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. Suppose $L : F(U) \to F(U)$ is a unary operator. If L satisfies the following axiom:*

$$L(\cap_{i\in I}(\lambda_i \vee X_i)) = \cap_{i\in I}(\lambda_i \vee L(X_i))$$

for any given index I, $X_i \in F(U)$ and $\lambda_i \in [0, 1]$, then there exists a unique fuzzy relation R on U such that for all $X \in F(U), L(X) = \underline{R}X$.

Proof. Using operator L, we construct a fuzzy relation R on U as follows:

$$R(x, y) = 1 - L(\{y\}')(x).$$

By using the axiom $L(\cap_{i\in I}(\lambda_i \vee X_i)) = \cap_{i\in I}(\lambda_i \vee L(X_i))$, for $x, y \in U$,

$$\begin{aligned} L(X)(x) &= L(\cap_{y\in U}(X(y) \vee \{y\}'))(x) = \wedge_{y\in U}(X(y) \vee L\{y\}'(x)) = \wedge_{y\in U}(X(y) \vee (L\{y\}')(x)) \\ &= \wedge_{y\in U}(X(y) \vee (1 - R(x, y))) = \underline{R}X(x). \end{aligned}$$

Similar to the case of crisp, R is unique. □

We need make full use of the outer product of two fuzzy sets.

Theorem 5.1. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. Suppose $L : F(U) \to F(U)$ is the unary operator, for any given index I, L satisfies $L(\cap_{i \in I}(\lambda_i \vee X_i)) = \cap_{i \in I}(\lambda_i \vee L(X_i))$ for all $i \in I$, $X_i \in F(U)$, and $\lambda_i \in [0, 1]$. We have the following three results.*

(1) *There exists a unique reflexive fuzzy relation R on U such that $L(X) = \underline{R}X$ for all fuzzy set X in U if and only if $L(X) \subseteq X$ for all $X \in F(U)$.*
(2) *There exists a unique symmetric fuzzy relation R on U such that $L(X) = \underline{R}X$ for all fuzzy set X in U if and only if $[X, L(Y)] = [Y, L(X)]$ for all $X, Y \in F(U)$.*
(3) *There exists a unique transitive fuzzy relation R on U such that $L(X) = \underline{R}X$ for all fuzzy set X in U if and only if $L(X) \subseteq L(L(X))$ for all $X \in F(U)$.*

Proof. The proof is similar to that of Theorem 3.1. □

Note that condition $L(\cap_{i \in I}(\lambda_i \vee X_i)) = \cap_{i \in I}(\lambda_i \vee L(X_i))$ can be derived from $[X, L(Y)] = [Y, L(X)]$, i.e., we have the following lemma.

Lemma 5.3. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power of U. If $L : F(U) \to F(U)$ is the unary operator that satisfies $[X, L(Y)] = [Y, L(X)]$, then $L(\cap_{i \in I}(\lambda_i \vee X_i)) = \cap_{i \in I}(\lambda_i \vee L(X_i))$ for any given index set I and $X_i \in F(U)$.*

Proof. Since, for any given index I,

$$\begin{aligned}
[Y, L(\cap_{i \in I}(\lambda_i \vee X_i))] &= [\cap_{i \in I}(\lambda_i \vee X_i), L(Y)] = \wedge_{i \in I}[\lambda_i \vee X_i, L(Y)] = \wedge_{i \in I}\lambda_i \vee [X_i, L(Y)] \\
&= \wedge_{i \in I}(\lambda_i \vee [X_i, L(Y)]) = \wedge_{i \in I}(\lambda_i \vee [Y, L(X_i)]) = \wedge_{i \in I}[Y, \lambda_i \vee L(X_i)] \\
&= [Y, \cap_{i \in I}(\lambda_i \vee L(X_i)])
\end{aligned}$$

for all $X_i, Y \in F(U)$, by the property of outer product, we have $L(\cap_{i \in I}(\lambda_i \vee X_i)) = \cap_{i \in I}(\lambda_i \vee L(X_i))$. □

Theorem 5.1 and Lemma 5.3 imply the following interesting corollary:

Corollary 5.1. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power of U. If $L : F(U) \to F(U)$ is the unary operator that satisfies $[X, L(Y)] = [Y, L(X)]$ for all $X \in F(U)$, then there exists a unique symmetric fuzzy relation R on U such that $L(X) = \underline{R}X$ for all $X \in F(U)$.*From Theorem 5.1 and Corollary 5.1, we obtain the axiomatization of the fuzzy rough set upper approximation operation over the arbitrary universal set.

Theorem 5.2. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. Suppose $L : F(U) \to F(U)$ is the unary operator that satisfies axioms*:

(1) $L(X) \subseteq X$ *for all* $X \in F(U)$;
(2) $[X, L(Y)] = [Y, L(X)]$ *for all* $X, Y \in F(U)$;
(3) $L(X) \subseteq L(L(X))$ *for all* $X \in F(U)$.

Then there exists a unique fuzzy similarity relation R on U such that $L(X) = \underline{R}X$ for each fuzzy set X in U. If we define the lower approximation operation using the formula, $\overline{R}X = -\underline{R}(-X)$, then $(\underline{R}X, \overline{R}X)$ is a fuzzy rough set. Moreover, axioms (1), (2), and (3) are consistent, necessary and independent each other.

Proof. By using Corollary 5.1, axiom (2) guarantees that there exists a symmetric fuzzy relation R on U such that $L(X) = \underline{R}X$ for each fuzzy set X in U. Using Theorem 5.1, axioms (1) and (3) guarantee that R is reflexive and transitive. Thus R is a fuzzy similarity relation on U. Similar to the case of crisp, axioms (1), (2), and (3) are consistent, necessary and independent each other. This completes the proof. □

Similar to the case of crisp, we have the following result.

Theorem 5.3. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. Suppose $L : F(U) \to F(U)$ is the unary operator that satisfies axioms*:

(1) $[X, L(Y)] = [Y, L(X)]$ *for all* $X, Y \in F(U)$;
(2) $X \cap L(L(X)) = L(X)$ *for all* $X \in F(U)$.

Then there exists a unique fuzzy similarity relation R on U such that $L(X) = \underline{R}X$ for each fuzzy set X in U. For the upper approximation, we have the following dual result.

Theorem 5.3′. *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. Suppose $H : F(U) \to F(U)$ is the unary operator that satisfies axioms*:

(1) $(X, H(Y)) = (Y, H(X))$ *for all* $X, Y \in F(U)$;
(2) $X \cup H(H(X)) = H(X)$ *for all* $X \in F(U)$.

Then there exists a unique fuzzy similarity relation R on U such that $H(X) = \overline{R}X$ for each fuzzy set X in U.

Axiomatic systems given in Theorem 5.3 and Theorem 5.3′ are much simpler than earlier works [10,15,22,23] on axiomatic approach because fuzzy lower approximation operations based on a fuzzy similarity relation can be characterized by two simple axioms in this paper. Moreover, Theorem 3.3 and Theorem 5.3 give a unified expression of the lower approximation operations axiomatic systems for both Pawlak's rough sets and fuzzy rough sets. Theorem 3.3′ and Theorem 5.3′ give a unified expression of the upper approximation operations axiomatic systems for both Pawlak's rough sets and fuzzy rough sets.

6. Conclusions

We used outer and inner products to characterize the lower and upper approximations of rough sets and fuzzy rough sets, respectively. We gave a unified and better expression for the axiomatic systems of lower and upper approximation operations for Pawlak's rough sets and fuzzy rough sets. The axiomatic approaches can help us to gain much more insight into the logical structure of rough sets and fuzzy rough sets. It also helps unify Pawlak's rough sets and fuzzy rough sets.

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